1 Consistency of TSQM with Standard Quantum Mechanics

ABL calculated the probability for the outcome of measuring any observable $\hat{A}$ (with eigenvalues a_j and eigenvectors $|a_j\rangle$) when this measurement occurred at a time t (which is intermediate to the pre- and post-selection $t \in [t_{\rm in}, t_{\rm fin}]$) and subject to obtaining the pre-and-post-selection:

$$Prob_{\rm ABL}(a_j, t|\Psi_{\rm in}, t_{\rm in}; \Psi_{\rm fin}, t_{\rm fin}) = \frac{|\langle\Psi_{\rm fin}|U_{t\to t_{\rm fin}}|a_j\rangle\langle a_j|U_{t_{\rm in}\to t}|\Psi_{\rm in}\rangle|^2}{\sum_n |\langle\Psi_{\rm fin}|U_{t\to t_{\rm fin}}|a_n\rangle\langle a_n|U_{t_{\rm in}\to t}|\Psi_{\rm in}\rangle|^2} \tag{1.1}$$

This is intuitive: we start with the pre-selection, $|\Psi_{\rm in}\rangle$. We then evolve this forward in time with the time displacement operator: $U_{t_{\rm in}\to t} = \exp\{-iH(t-t_{\rm in})\}$ (figure 2.a, H the free Hamiltonian). Then $|\langle a_j|U_{t_{\rm in}\to t}|\Psi_{\rm in}\rangle|^2$ gives us the probability to obtain $|a_j\rangle$. We then evolve $|a_j\rangle$ forward in time with $U_{t\to t_{\rm fin}}$. Then $|\langle\Psi_{\rm fin}|U_{t\to t_{\rm fin}}|a_j\rangle|^2$ is the probability to obtain $|\Psi_{\rm fin}\rangle$. The total conditional probability to obtain $|a_j\rangle$ given all three stages is the product, eq. 1.1, the denominator giving the normalization.

Consider the time-reverse of eq. 1.1: $\langle U_{t_{\rm fin}\to t}\Psi_{\rm fin}|a_j\rangle\langle U_{t\to t_{\rm in}}a_j|\Psi_{\rm in}\rangle$ (figure 2.b). This is obtained by applying $U_{t\to t_{\rm fin}}$ on $\langle\Psi_{\rm fin}|$ **instead** of on $\langle a_j|$ and $U_{t_{\rm in}\to t}$ on $\langle a_j|$ instead of on $|\Psi_{\rm in}\rangle$. (Note: re-writing $\langle\Psi_{\rm fin}|U_{t\to t_{\rm fin}}$ as $\langle U^\dagger_{t\to t_{\rm fin}}\Psi_{\rm fin}|$ occurs throughout quantum mechanics due to the symmetry $U^\dagger_{t\to t_{\rm fin}} = \left\{e^{-iH(t_{\rm fin}-t)}\right\}^\dagger = e^{iH(t_{\rm fin}-t)} = e^{-iH(t-t_{\rm fin})} = U_{t_{\rm fin}\to t}$.)

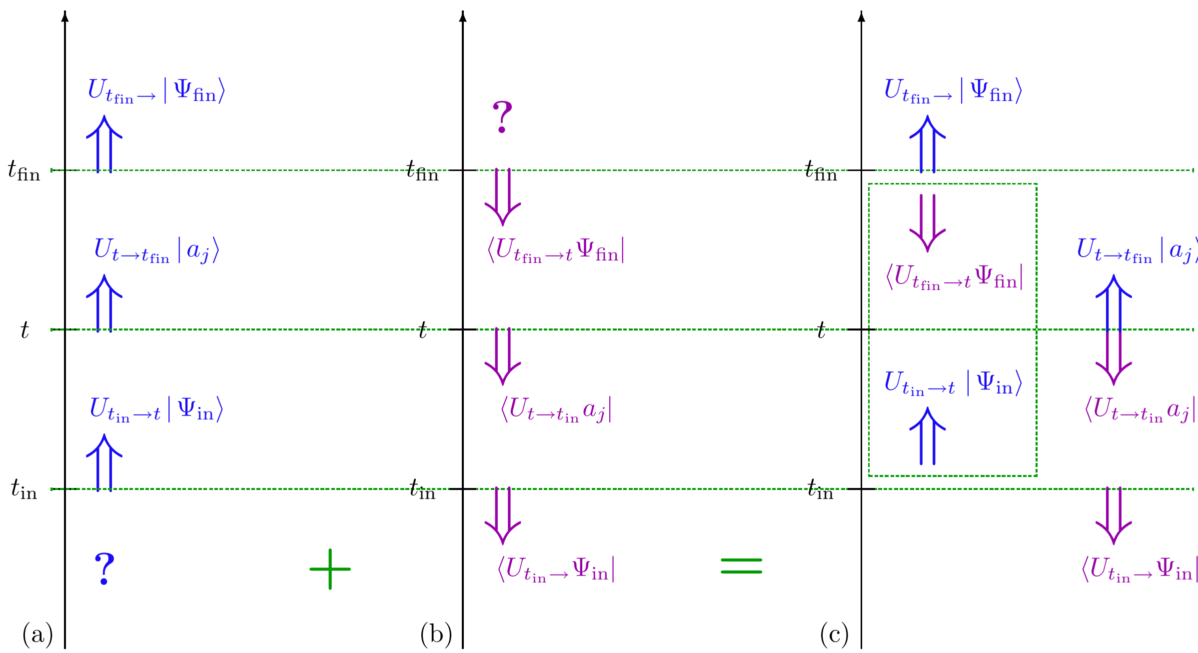

Figure 2: Time-reversal symmetry in probability amplitudes.

We note that the 2-vectors relevant to the intermediate time t, are $\langle U_{t_{\rm fin}\to t}\Psi_{\rm fin}|$ and $U_{t_{\rm in}\to t}\ |\Psi_{\rm in}\rangle$ (which are not just the time-reverse of each other). So, neither figures 2.a nor 2.b by themselves give the right picture for measurements at t. In other words, we need $\langle U_{t_{\rm fin}\to t}\Psi_{\rm fin}|$ which propagates the post-selection back to t, and $U_{t_{\rm in}\to t}\ |\Psi_{\rm in}\rangle$ which propagates the pre-selection forward to t, (see conjunction of both figures 2.a and 2.b giving 2.c). This simple mathematical manipulation leads to the main idea behind TSQM:

$$Prob_{\rm ABL}(a_j, t|\Psi_{\rm in}, t_{\rm in}; \Psi_{\rm fin}, t_{\rm fin}) = \frac{|\langle U_{t_{\rm fin}\to t}\Psi_{\rm fin}|a_j\rangle\langle a_j|U_{t_{\rm in}\to t}|\Psi_{\rm in}\rangle|^2}{\sum_n |\langle U_{t_{\rm fin}\to t}\Psi_{\rm fin}|a_n\rangle\langle a_n|U_{t_{\rm in}\to t}|\Psi_{\rm in}\rangle|^2} \tag{1.2}$$

We emphasize several advantages evident even from this step:

- TSQM is consistent with standard quantum mechanics.
- the probability (and amplitude) are symmetric under exchange of $|\Psi_{\rm in}\rangle$ and $|\Psi_{\rm fin}\rangle$, and,
- whenever we encounter a bra, it suggests a state is propagating backwards-in-time.
- Evaluating time-asymmetric ABL requires more computation: many time evolutions in the denominator of eq. 1.1, $U_{t\to t_{\rm fin}}|a_1\rangle$... $U_{t\to t_{\rm fin}}|a_n\rangle$, in contrast to a single time evolution in eq. 1.2 $\langle U_{t_{\rm fin}\to t}\Psi_{\rm fin}|$ (see §3).

1.1 A paradox

We motivate TSQM with a paradox concerning the relativistic covariance of the state-description.

Consider two spin-1/2 particles prepared in an EPR state: $|\Psi_{EPR} t = 0)\rangle = \frac{1}{\sqrt{2}}\{|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\}$. Suppose at some later time t_2, the particles separate to a distance L and **A**lice measures her spin in the z-direction and obtains the outcome $|\uparrow_z\rangle_A$. According to the usual interpretation, measurements on either particle will instantly reduce the EPR state into a direct product $|\Psi(t_2+\varepsilon) = |\uparrow_z\rangle_A |\downarrow_z\rangle_B$. **I.e. after A**lice performs her measurement at $t = t_2$, then the joint wavefunction collapses so when **B**ob measures his particle at $t_2 + \varepsilon$, he will find that it had collapsed to $|\downarrow_z\rangle_B$ (see figure 3.a).

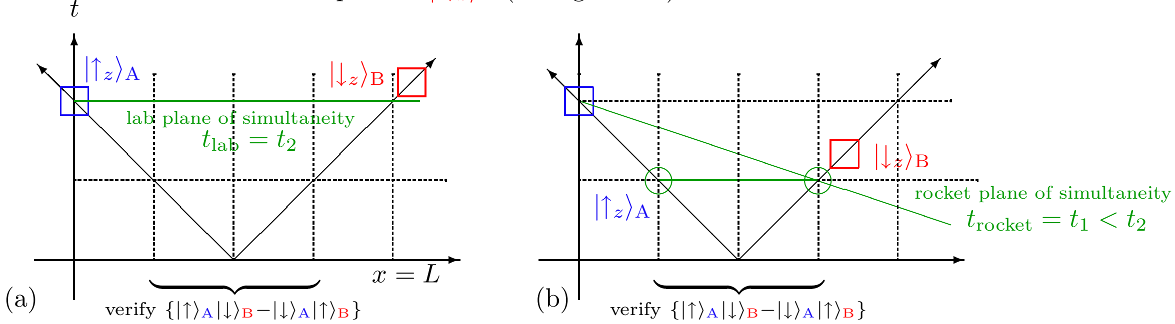

Figure 3: Collapse of singlet state in 2-different frames of reference; a) the $t_{lab} = 0$ hypersurface intersects the B worldline before B's measurement, b) the $t_{rocket} = 0$ hypersurface intersects the B worldline *after* B's measurement.

However, in a rocket-frame, **B**ob's measurement occurs first and then **A**lice's (see figure 3.b). The lab-frame believes that **A**lice's measurement caused the collapse whereas the rocket-frame disagrees and believes that **B**ob's measurement caused the collapse. While the two different versions give the same statistical results at the level of probabilities, they differ completely on the description of the state during the intermediate times.

TSQM re-introduces Lorentz covariance because we must always look to a future vector in order to complete the specification of the state. The boundary condition of the other particle always enforces correlations, without the need to specify a moment when it becomes true due to some non-local state reduction.

In our example, the full state is the bra-ket:

$$\langle\Psi_{fin}||\Psi_{EPR}\rangle = \frac{1}{\sqrt{2}}\langle\uparrow_z|_A \{\langle\uparrow_B + \langle\downarrow|_B\} \frac{1}{\sqrt{2}}\{|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\} \tag{1.3}$$

Both $|\uparrow_z\rangle_A$ and $|\Psi_{EPR}\rangle$ were determined by **local** measurements and therefore transform covariantly as do all local events. If **A**lice changes her mind and measures the y-direction instead of the z-direction, then this would change the post-selected vector all the way back to $|\Psi_{EPR}\rangle$. This solves the paradox.[**?**].

1.2 Pre-and-post-selection and Spin-1/2

TSQM reveals surprising properties even for the simplest quantum system. Consider a spin-1/2 particle pre-selected $|\Psi_{\rm in}\rangle = |\hat{\sigma}_x = +1\rangle = |\uparrow_x\rangle$ and post-selected $|\Psi_{\rm fin}\rangle = |\hat{\sigma}_{\rm y} = +1\rangle = |\uparrow_y\rangle$, so half the particles are excluded. ABL, eq. 1.2, tells us the outcome for ideal-measurements during $t \in [t_{\rm in}, t_{\rm fin}]$: if $\hat{\sigma}_x$ is measured then:

$$Prob_{\rm ABL}(\hat{\sigma}_x = +1) = 1 \tag{1.4}$$

If $\hat{\sigma}_y$ is measured, then

$$Prob_{\rm ABL}(\hat{\sigma}_y = +1) = 1 \tag{1.5}$$

One may legitimately wonder, however, if this isn't all quite trivial. Indeed, such pre and post-selected ensembles are, of course, quite common in classical (i.e. non quantum) context. In quantum mechanics, however, there is no way, *even in principle*, to know the result of the final measurement before t_1. Post-selection is thus not a surrogate for a more careful pre-selection. On the contrary, it leads to genuinely new information.

Consider measuring the spin in a direction $\xi = 45°$ relative to the $x - y$ axis:

$$\hat{\sigma}_{45°} = \hat{\sigma}_x \cos 45° + \hat{\sigma}_y \sin 45° = \frac{\hat{\sigma}_x + \hat{\sigma}_y}{\sqrt{2}} \tag{1.6}$$

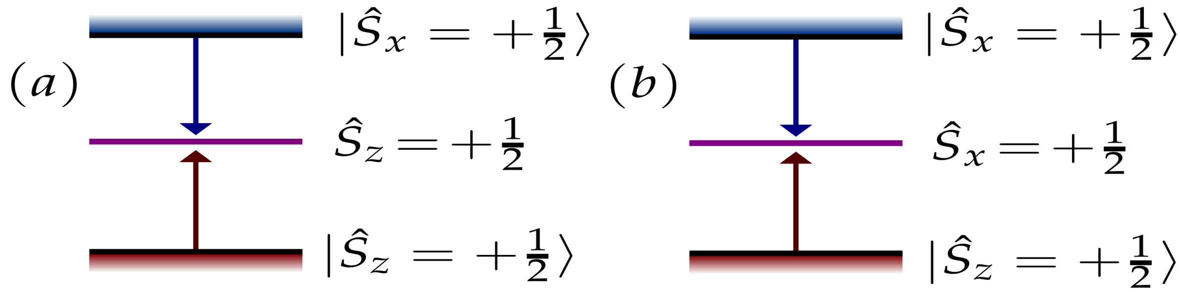

Figure 4: A spin-1/2 particle in a region free of external magnetic fields is pre-selected at time t_0 to be in the quantum state with spin up in the z direction, and post-selected at t_1 to be in the state with spin up in the x direction. At any intermediate time, such a particle would have well-defined values of the two non-commuting spin components S_z and S_x. Both would have to be +1/2.

If we insert both values, eq. 1.4, $\hat{\sigma}_{\rm x} = +1$, **and** eq. 1.5, $\hat{\sigma}_{\rm y} = +1$, into eq. 1.6, then we would obtain

$$\hat{\sigma}_{45^\circ} = \frac{1+1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \tag{1.7}$$

. But, that can't be right since the eigenvalues for any spin operator, including $\hat{\sigma}_\xi$, must be ± 1. Moreover, the predicted value $\hat{\sigma}_{45^\circ} = \sqrt{2}$ is even larger than the largest possible eigenvalue, +1.

Still, Aharonov's intuition was that there should be some way for both $Prob_{\rm ABL}(\hat{\sigma}_x = +1) = 1$ and $Prob_{\rm ABL}(\hat{\sigma}_y = +1) = 1$ to manifest simultaneously to produce $\hat{\sigma}_\xi = \sqrt{2}$. The following suggests a clue. To measure $\hat{\sigma}_\xi$ say with a Stern-Gerlach apparatus, we switch on a magnetic field in the x-y direction. This magnetic field starts to precess the pre-selected $\hat{\sigma}_x$ and the post-selected $\hat{\sigma}_y$ around the x-y direction. This also follows from the fact that $\hat{\sigma}_x$ and $\hat{\sigma}_y$ don't commute. Obviously as $[\hat{\sigma}_y, \hat{\sigma}_x] \neq 0$, the measurement of $\hat{\sigma}_y$ completely disturbs $\hat{\sigma}_x$ and vice-versa. Since we disturb the two-vectors, we should not expect that both vectors could be relevant to the intermediate time.

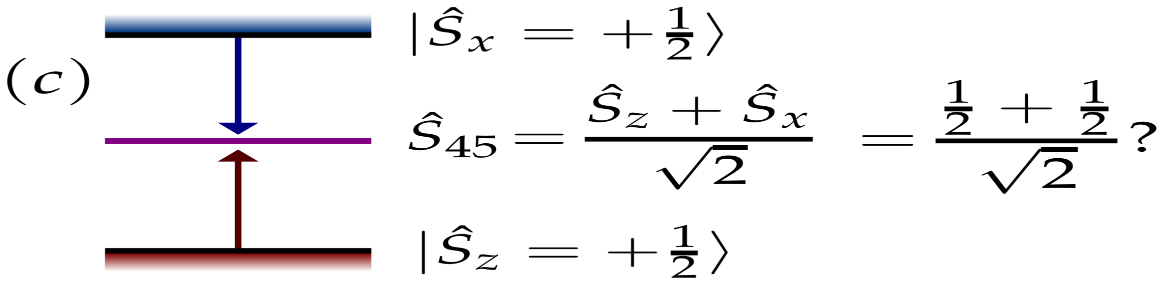

Figure 5: It would seem to follow that the spin component $S_{\frac{\pi}{4}}$ along the 45° direction in the xz plane would have to be $+\sqrt{2}/2$, not an eigenvalue and seemingly impermissible.