# 2 Interference phenomenon from the Heisenberg perspective: modular variables

As we have argued previously, the basic gauge symmetry would be violated if any quantum experiment could measure the local phase in $|\Psi_\alpha\rangle$ and therefore there is no *locally accessible phase* information in $|\Psi_\alpha\rangle$. The relative phase is a truly *non-local* feature of quantum mechanics. This point is often missed when the Schrödinger picture is taught and classical intuitions are applied to interference. For this and other reasons, we maintain that the non-local aspect of interference is clearer in the Heisenberg picture.

## 2.1 Modular variables are the observables that are sensitive to the relative phase

In §II.b, we pointed to the significance of the Heisenberg translation operator, $\exp\{\pm\frac{i}{h}\hat{p}D\}$, effecting $\exp\{-\frac{i}{h}\hat{p}D\}\psi_{\rm R}(x) \to \psi_{\rm R}(x-D)$ overlapping with $\psi_{\rm L}(x)$. Therefore, the expectation value of the translation operator $\exp\{\frac{i}{h}\hat{p}D\}$ ***does*** depend on $\alpha$: $\langle\Psi_\alpha \mid \exp\{i\hat{p}D/h\} \mid \Psi_\alpha\rangle = e^{-i\alpha}/2$.

But, exactly *what* information about $\alpha$ does $\langle\exp\{\pm\frac{i}{h}\hat{p}D\}\rangle$ reveal? It is easy to see that if we replace $p$ with $p-\frac{nh}{D}$ ($n$ is the largest integer such that $n\frac{h}{D} < p$ (i.e. satisfying $0 \le \hat{p} - n\frac{h}{D} \le \frac{h}{D}$), then $e^{\frac{i}{\hbar}\hat{p}D}$ changes by $e^{\frac{iD}{\hbar}\frac{nh}{D}} = e^{in2\pi} = 1$, i.e. nothing changes. This means that $e^{\frac{i}{\hbar}\hat{p}D}$ gives us information about the remainder after this integer number of $\frac{h}{D}$ is subtracted from $p$. This is otherwise known of as the modular momentum $p_{\rm mod} \equiv \hat{p}$ modulo $\frac{h}{D}$ (see fig. 2) defined by: $\hat{p}$ modulo $\frac{h}{D} \equiv \hat{p} - n\frac{h}{D}$.

It is clear that $p$ *mod* $\frac{h}{D}$ has the topology of a circle, like any periodic function. Every point on the circle is another possible value for $p_{\rm mod}$. We deal with modular quantities every time we look at a wristwatch which displays the time modulo 12.

We can get back to ordinary momentum through the relation:

$$p = N_p\frac{h}{D} + p_{\rm mod} \tag{3}$$

We can see this (fig 2) if we stack an integer number ($N_p$) of $\frac{h}{D}$ on top of the modular portion of $p$ ($p_{\rm mod}$ is the lower portion of fig 2). Note that the eigenstates of the translation operator $\exp\{\frac{i}{\hbar}\hat{p}D\}$ are also eigenstates of the modular momentum $p_{\rm mod}$.

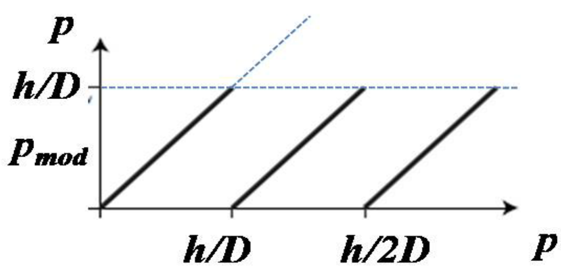


Figure 2: Stacking an integer number ($N_p$) of $\frac{h}{D}$ on top of the modular portion of $p$ ($p_{\rm mod}$.

## 2.2 For interference phenomenon, modular variables satisfy non-local equations of motion

The key to our explanation of interference from the single particle perspective are the non-local equations of motion satisfied by these modular variables. Thus, using $H = \frac{p^2}{2m} + V(x)$ and $e^{\frac{i}{\hbar}\hat{p}D}V(x)e^{-\frac{i}{\hbar}\hat{p}D} = V(x+D)$, we find **non-local** [12, 10] Heisenberg equations of motion for modular variables:

$$\frac{d}{dt}e^{\frac{i}{\hbar}\hat{p}D} = \frac{i}{\hbar}[H, e^{\frac{i}{\hbar}\hat{p}D}] = \frac{i}{\hbar}[V(x) - V(x+D)]e^{\frac{i}{\hbar}\hat{p}D} \tag{4}$$

with $e^{\frac{i}{\hbar}\hat{p}D}$ changing even when $\frac{\partial V}{\partial x} = 0$.

This, essentially quantum phenomenon, has no classical counterpart. The classical equations of motion for any function $f(p)$ derives from the Poisson bracket:

$$\frac{df(p)}{dt} = \{f(p), H\}_{PB} = -\frac{\partial f}{\partial p}\frac{\partial H}{\partial x} + \underbrace{\frac{\partial f}{\partial x}}_{=0}\frac{\partial H}{\partial p} = 0 \tag{5}$$

i.e. $f(p)$ changes only if $\frac{\partial V}{\partial x} \neq 0$ at the particle's location.

Unlike the Poisson bracket in classical mechanics, quantum mechanics has non-trivial and unique solutions to the commutator: $[f(p), g(x)] = 0$ if $f(p) = f(p + p_o)$, $g(x) = g(x + x_o)$ and $x_o p_o = h$. This leads us to a new structure within quantum mechanics of periodic functions which lead naturally to the concept of modular variables. In our particular case, $x_o = D$, $p_o = \frac{h}{D}$ and $f(p) = e^{\frac{i}{\hbar}\hat{p}D}$, so $g(x)$ only depends on the function $g(x)$ modulu $D$ and $f(p)$ only depends on the function $f(p)$ modulu $\frac{h}{D}$. The non-local equations of motion that $e^{\frac{i}{\hbar}\hat{p}D}$ satisfies show how the potential at the left slit *does* affect the evolution of the modular variable even when we consider a particle located at the right slit (and vice-versa, see fig. 3). Modular variables obey non-local equations of motion independent of the specific state of the Schrödinger wavefunction, whether it is localized around one slit or in a superposition. Nevertheless, the modular momentum may change (non-locally) even if the wavefunction experiences no force. We can therefore see that the non-local effect of the open or closed slit is to produce a shift in the modular momentum of the particle while leaving the expectation values of moments of its momentum unaltered.

### 2.3 Non-local exchange of modular variables in the double-slit setup

For the special case of two-slits, a set of spin-like observables can be identified as members of the set of deterministic operators. For simplicity (without affecting the generality of our arguments), we can express the relevant modular variable as the parity (exchange) operation $\hat{\mathsf{P}}$ (effecting $\hat{\mathsf{P}}|\psi_{\mathrm{L}}\rangle = |\psi_{\mathrm{R}}\rangle$ and $\hat{\mathsf{P}}|\psi_{\mathrm{R}}\rangle = |\psi_{\mathrm{L}}\rangle$). It **is** sensitive to the relative phase $\alpha$ between the disjoint lumps of eq. 1 [10] [12] [31]:

$$\begin{aligned} \langle\Psi_\alpha|\hat{\mathsf{P}}|\Psi_\alpha\rangle &= \frac{1}{2}\left\{\langle\psi_{\mathrm{L}}| + e^{-i\alpha}\langle\psi_{\mathrm{R}}|\right\}\hat{\mathsf{P}}\left\{|\psi_{\mathrm{L}}\rangle + e^{i\alpha}|\psi_{\mathrm{R}}\rangle\right\} \\ &= \frac{1}{2}\left\{e^{i\alpha} + e^{-i\alpha}\right\} = \langle\cos\alpha\rangle \end{aligned} \tag{6}$$

To simplify further, we will focus on the $\pm 1$ eigenstates of $\hat{\mathsf{P}}$: $\psi_{\mathrm{L}}(x) + \psi_{\mathrm{R}}(x)$ and $\psi_{\mathrm{L}}(x) - \psi_{\mathrm{R}}(x)$. A measurement of which slit the particle goes through (i.e. a WWM) will change the value of $\langle\hat{\mathsf{P}}\rangle$. For example if the initial state is $|\psi_{\mathrm{L}}\rangle + |\psi_{\mathrm{R}}\rangle$, then $\langle\cos\alpha\rangle = 1$, i.e. $\langle\hat{\mathsf{P}}\rangle = 1$. If we collapse the state to $|\psi_{\mathrm{R}}\rangle$, then $\langle\cos\alpha\rangle = 0$ and $\langle\hat{\mathsf{P}}\rangle = 0$.

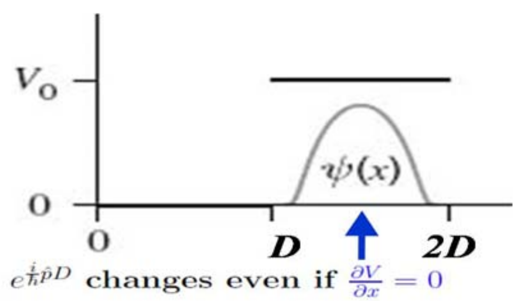


Figure 3: A potential with 2 values and a wave-packet with support only in the interval $D < x < DL$.

We can also see from eq. 4 that if the left slit is open, then $V(x) - V(x + D) = 0$, and therefore $p_{\mathrm{mod}}$ is conserved. However, if the left slit is closed, then $V(x) - V(x + D) \neq 0$ and $p_{\mathrm{mod}}$ is not conserved.

### 2.4 Why does the interference pattern disappear when the particle is localized?

When we obtain WWM information, we collapse the superposition from $|\Psi_\alpha\rangle$ to $|\psi_{\mathrm{L}}\rangle$ or $|\psi_{\mathrm{R}}\rangle$ (in the Schrödinger picture). In the Heisenberg picture, however, we cannot describe the collapse of a superposition. The wavefunction is still of course relevant as a boundary condition, but it does not evolve in time. Only the operators evolve in time according to the Heisenberg equation of motion: $\frac{dA_H}{dt} = \frac{i}{\hbar}[H, A_H] + U^{-1}(t)\frac{\partial A_s}{\partial t}U(t)$. But which operators become uncertain when WWM information is obtained?

Suppose again the particle travels through the right slit and we choose either to open or close the left slit. This action causes a non-local exchange of modular momentum between the potential at the left slit and the particle going through the right slit. Is this observable?

Up until [1], it was believed that this could not be observed. The reason is that modular momentum (unlike ordinary momentum) becomes, upon detecting (or failing to detect) the particle at a particular slit, *maximally* uncertain. In other words, the effect of introducing a potential at a distance $D$ from the particle (i.e. of opening a slit) is equivalent to a rotation in the space of the modular variable - let's call it $\theta$ - that is exchanged nonlocally. Suppose the amount of nonlocal exchange is given by $\delta\theta$ (i.e. $\theta \to \theta + \delta\theta$). Now "maximal uncertainty" means that the probability to find a given value of $\theta$ is independent of $\theta$, i.e. $P(\theta) = constant = \frac{1}{2\pi}$. Under these circumstances, the shift in $\theta$ to $\theta + \delta\theta$ will introduce no observable effect, since the probability to measure a given value of $\theta$, say $\theta_1$, will be the same before and after the shift, $P(\theta_1) = P(\theta_1 + \delta\theta_1)$. We shall call a variable that satisfies this condition a "completely uncertain variable". Using this, APP proved a stronger *qualitative* uncertainty principle for the modular momentum, instead of the usual quantitative statement of the uncertainty principle (e.g. $\Delta p D \geq \hbar$): if the nonlocal exchange of any modular variable $\theta$ came close to violating causality, then the probability distribution for all averages of that modular variable flattens out, i.e every value for $\theta$ became equally probable and change in $\theta$ becomes un-measureable:

**Theorem II qualitative uncertainty principle for modular variables**: if $\langle e^{in\theta}\rangle = 0$ for any integer $n \neq 0$ and if $\theta$ is a periodic function with period $\tau$, then $\theta$ is completely uncertain if $\theta$ is uniformly distributed on the unit circle.
Proof: we expand the probability density $Prob(\theta)$ to a Fourier series $Prob(\theta) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}$ (integer $n$ is a requirement for the function to be periodic in $\theta$), where $a_n = \int Prob(\theta) e^{in\theta} d\theta = \langle e^{in\theta}\rangle$ (since the average of any function is given the integral of the function with the probability). We see that $Prob(\theta) = const$ if and only if $a_n = 0$ for all $n \neq 0$, and therefore $\langle e^{in\theta}\rangle = 0$ for $n \neq 0$.

Consider how this works in the double slit setup. Let us start with a particular $|\Psi_\alpha\rangle$, namely the symmetric ($\alpha = 0$, $|\psi_{\rm L}\rangle + |\psi_{\rm R}\rangle$) or anti-symmetric ($\alpha = \pi$, $|\psi_{\rm L}\rangle - |\psi_{\rm R}\rangle$) states. The parity $\hat{\mathsf{P}}$ then has sharp eigenvalues $\pm 1$. However, $\hat{\mathsf{P}}$ becomes maximally uncertain when the state is localized at one slit: by definition, $\overline{\hat{\mathsf{P}}^2} = 1$, however, $\langle\psi_{\rm L}|\hat{\mathsf{P}}|\psi_{\rm L}\rangle$=$\langle\psi_{\rm L}|\psi_{\rm R}\rangle$=0 so $\overline{\hat{\mathsf{P}}}$=0, and therefore $\Delta\hat{\mathsf{P}} \equiv \sqrt{\overline{(\hat{\mathsf{P}})}^2 - \overline{\hat{\mathsf{P}}}^2}$ =1, i.e. we have maximal uncertainty when the particle is localized at one slit. Stated differently, when the particle is at the right (or left) slit its' wave function is a superposition with *equal* weights of the two parity eigenstates $|\psi_{\rm L}\rangle \pm |\psi_{\rm R}\rangle$ with $\pm 1$ eigenvalues which by definition is the state of maximal variance of the operator involved.[3]

The vanishing of the expectation value of the modular momentum variable (per Theorem II) is the manifestation in our present picture of the loss of information on $\alpha$ and of the interference pattern, once we localize the particle at the left or right slit. [4]

This brings us to what we believe to be a more physical answer (from the perspective of an individual particle) for the disappearance of interference: the momentum exchange with the left slit and resulting momentum uncertainty (destroying the interference pattern when the left slit is closed) is not that of ordinary momentum since as we noted $\Delta p$ does not change. Rather, the closing of the left slit and localization of the particle at the right slit involves a non-local exchange of *modular* momentum. This phenomenon can also be demonstrated for any refinement of the double-slit. For example, any measurement at the left slit introduces an uncertain potential there. As a result of the non-local equations of motion, this introduces complete uncertainty in the modular variable. Thus, detecting which slit a particle passes through destroys all information about the modular momentum.

It therefore appears that no observable effect of one slit acting on the particle traveling through the other slit can be obtained via the nonlocal equations of motion of the modular variable and therefore, this non-locality "peacefully co-exists" with causality. Have we not violated the dictum of maintaining the closest correspondence between measurement and theory by claiming the existence of a new kind of nonlocal - yet un-observable - effect?

The key novel observation we next make is that this non-locality *does have* an observable meaning in the context of weak measurements on pre- and post-selected ensembles [15] [18]

[3] In passing, we note that this is readily extended from the $Z(2)$ case of just two slit to the $Z(N)$ case of $N$ equidistant, equal slits with periodic boundary conditions (see Appendix B).

[4] Although much of the discussion in this article focuses on the simplest interference example with 2-slits, our approach becomes clearer when it is applied to an infinite number of slits. See Appendix C.